MATHEMATICAL ASPECTS OF QUANTUM THEORY

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BIMSA QUANTUM TEAM

Abstract. To foster dialogue and collaboration among young researchers, the BIMSA quantum team has initiated a call for young mathematical experts to present problems linked to quantum theory. This collection of notes comprises challenges in operator theory, noncommutative analysis, vertex operator algebras, conformal field theory, probability theory and quantum information. These thought-provoking problems have been contributed by speakers and participants who attended the "Mathematical Aspects of Quantum Theory" conference in Sanya, Hainan, from January 12 to January 17, 2024, co-hosted by Tsinghua University and BIMSA.

Contents

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1. Problems on Chiral de Rham Complex (**Contributed by Xuanzhong Dai**)

Chiral de Rham complex introduced by Malikov et al. in 1998, is a sheaf of topological vertex algebras on any complex analytic manifold or non-singular algebraic variety X [MSV]. The sections of the chiral de Rham complex on each affine open chart is isomorphic to $\dim X$ copies of $\beta\gamma - bc$ system. Starting from the vertex algebra of global sections of chiral de Rham complex on the upper half plane, we consider the subspace of Γ-invariant sections that are meromorphic at the cusps. The space is again a vertex operator algebra, with a linear basis consisting of lifting formulas of meromorphic modular forms. As the fractional linear transformation on $\mathbb{H}$ induces an $SL(2,\mathbb{R})$-action on the chiral de Rham complex, we consider the vertex algebra of Γ-invariant global sections that are meromorphic at the cusps, denoted by $\mathcal{M}(\mathbb{H},\Gamma)$. It is surprising to see that the Γ-invariant vertex algebra is simple regardless of congruence subgroup Γ [DS].

The Rankin-Cohen bracket is a family of bilinear operations, which sends two modular forms f of weight k and h of weight l, to a modular form $[f,h]_n$ of weight $k+l+2n$. Let $\Gamma \subset SL(2,\mathbb{Z})$ be a congruence subgroup, and $f \in M_k(\Gamma)$, $h \in M_l(\Gamma)$, then the n-th Rankin-Cohen bracket is given by

$$[f,h]_n = \frac{1}{(2\pi i)^n} \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} f^{(r)}(\tau)h^{(s)}(\tau).$$

It was speculated by W. Eholzer, Y. Manin and D. Zagier long time ago that the Rankin-Cohen brackets are related to vertex operator algebras [Z]. Recently we give a precise formulation of the nontrivial relation in the sense that the vertex operations are totally determined by the modified Rankin-Cohen bracket in [D]. Note that the Rankin-Cohen bracket exhibits certain decay property in the presence of constant functions and thus we modify the Rankin-Cohen brackets specifically when a constant modular form is involved. It is shown in [NSZ] that the modified Rankin-Cohen brackets also appear in the description of modular linear differential operators.

Problem 1.1. *How to construct irreducible modules of $\mathcal{M}(\mathbb{H},\Gamma)$?*

Problem 1.2. *Can we obtain some new operators on modular forms from vertex operations?*

Problem 1.3. *How to relate the axioms of the Rankin-Cohen brackets with those of vertex operator algebras?*

References

[D] X.Dai, Chiral de Rham complex on the upper half plane and modular forms, Int. Math. Res. Not. (IMRN), (2022), no. 24, 19258-19299.

[DS] X.Dai and B. Song, Simple Vertex Algebras Arising From Congruence Subgroups, arXiv:2211.10090.

[NSZ] K. Nagatomo, Y. Sakai, and D. Zagier, Modular Linear Differential Operators and Generalized Rankin-Cohen Brackets, arXiv:2210.10686.

[MSV] F. Malikov, V. Schectman, and A. Vaintrob, Chiral de Rham complex. Comm. Math. Phys. 204 (1999), no. 2, 439-473.

[Z] D. Zagier, Modular forms and differential operators. K. G. Ramanathan memorial issue. Proc. Indian Acad. Sci. Math. Sci. 104 (1994), no. 1, 57-75.

2. Problems on Noncommtuative Carleson's theorem (**Contributed by Guixiang Hong**)

The Lusin conjecture, namely the partial summations of trigonometric series of any square integrable function on the unit circle should converge almost everywhere, had ever been the most famous open problem in Fourier analysis. This conjecture was posed by Luzin in 1913 after Kolmogorov's counterexample for integrable functions, and had been resolved by Carleson in 1966 [1]. Carleson obtained the pointwise convergence by establishing a maximal inequality, that is the L_2-boundedness of Carleson's maximal operator. An alternative proof of Carleson's theorem was provided by Fefferman [2], pioneering a set of ideas called time-frequency analysis. Lacey and Thiele [11] provided the first independent proof on the line of the boundedness of Carleson's maximal operator, which improves in some ways that of Fefferman's [2], by which it was inspired. The proof of Lacey and Thiele was a byproduct of their work [9, 10] on the boundedness of the bilinear Hilbert transforms. Together with other techniques such as transference principle, the time-frequency analysis has been used to study for instance ergodic theory, and thus nowdays it still play an important role in harmonic analysis and beyond.

On the other hand, noncommtuative martingale theory and noncommtuative analysis has gained rapid development since the seminal work due to Pisier, Xu and Junge on noncommutative Burkholder-Gundy and Doob's inequalities [12, 7]. In the last two decades, there appear a series of breakthrough work such as the theory of BMO spaces, Calderón-Zygmund operators and Fourier-Shur multipliers etc.. In particular, several maximal inequalities such as Dunford-Schwartz's maximal ergodic theorem [8], maximal ergodic theorem for actions of groups of polynomial growth [3, 5], pointwise convergence of noncommutative Fourier series [6] and noncommutative maximal inequalities for Calderón-Zygmund operators [4] have been successfully built in this new framework. However, whether there holds a noncommutative version of Carleson's theorem is an open problem circulated in the noncommutative analysis community, even though many tools and ideas have been developed in this setting as above.

Let us introduce some notations and formulate the problem explicitly. Let $(\mathcal{M}, \tau)$ be a noncommutative measure space equipped with a normal semifinite tracial state τ, and $S_{\mathcal{M}}$ be the weak-$*$ dense ideal of $\mathcal{M}$ with elements of finite trace support. We refer the reader to [13] for the resulting noncommutative L_p spaces. Given a $S_{\mathcal{M}}$-valued integrable function f on the unit circle $\mathbb{T}$, the Dirichlet summation method is defined as

$$(D_N f)(z) = \sum_{k=-N}^{N} \hat{f}(k) z^k, \quad z \in \mathbb{T}, \quad N \in \mathbb{N},$$

where $\hat{f}$ denotes the Fourier transform of f.

When $\mathcal{M} = \mathbb{C}$, Carlson's theorem states that $D_N f \to f$ almost everywhere as $N \to \infty$.

Conjecture. Let $f \in L_2(\mathbb{T}; L_2(\mathcal{M}))$, then $D_N f \to f$ bilaterally almost uniformly as $N \to \infty$. More precisely, given any $\varepsilon > 0$, there exist a projection e in $L_\infty(\mathbb{T})\overline{\otimes}\mathcal{M}$ such that $\tau \int_{\mathbb{T}}(1-e) < \varepsilon$ and $\|e(D_N f - f)e\|_\infty \to 0$.

References

[1] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135-157.

[2] C. Fefferman, Pointwise convergence of Fourier series, Ann. of Math. 98 (1973), 551-571.

[3] G. Hong, B. Liao, S. Wang, Noncommutative maximal ergodic inequalities associated with doubling conditions, Duke Math. J., 170 (2021), 205-246.

[4] G. Hong, X. Lai and B. Xu, Maximal singular integral operators acting on noncommutative L_p-spaces, Math. Ann. 386 (2023), no. 1-2, 375-414.

[5] G. Hong, S. Ray and S. Wang, Maximal ergodic inequalities for some positive operators on noncommutative L_p-spaces, J. London. Math. Soc., 108 (2023), no. 1, 362-408.

[6] G. Hong, S. Wang and W. Wang, Pointiwse convergence of noncommutative Fourier series, Memoirs AMS, in press.

[7] M. Junge, Doob's inequality for noncommutative martingale. J. Reine Angew. Math. 549 (2002), 149-190.

[8] M. Junge, Q. Xu, Noncommutative maximaler egodic theorems, J. Am. Math. Soc. 20 (2007), 385-439.

[9] M. T. Lacey and C. M. Thiele, L^p bounds for the bilinear Hilbert transform, $p > 2$, Ann. of Math. 146 (1997), 693-724.

[10] M. Lacey and C. Thiele, On Calderón's conjecture, Ann. of Math. 149 (1999), 475-496.

[11] M. Lacey and C. Thiele, A proof of boundedness of the Carleson operator, Math. Res. Lett. 7 (2000), 361-370.

[12] Pisier, G., Xu, Q.: Non-commutative martingale inequalities. Commun. Math. Phys. 189, 667-698 (1997)

[13] G. Pisier and Q. Xu, Non-commutative L_p-spaces, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1459-1517.

3. Problems on ISR Property (**Contributed by Yongle Jiang**)

Let Γ be a lattice in a higher-rank simple Lie group G with trivial center, e.g. $\Gamma = SL_3(\mathbb{Z}), G = SL_3(\mathbb{R})$. Recall that the celebrated Margulis' Normal Subgroup Theorem says that every normal subgroup in Γ is either trivial or of finite index.

In [1], Alekseev and Brugger considered a natural generalization of this to the group von Neumann algebra setting by asking whether every regular subfactor P of $L(\Gamma)$ is trivial or of finite index. Here P is regular if the normalizer of P, i.e. those unitaries $u \in L(\Gamma)$ such that $uPu^* = P$, generates $L(\Gamma)$ as a von Neumann algebra. In practice, we further assume that Γ is contained in the normalizer of P, i.e. P is Γ-invariant. In [5], Kalantar and Panagopoulos proved that for the above Γ, every Γ-invariant von Neumann subalgebra is of the form $L(\Lambda)$ for some normal subgroup $\Lambda \lhd \Gamma$. This motivated us to introduce the following notion in [2].

Definition 3.1. *Let G be a countable discrete group. We say G has the invariant von Neumann subalgebras rigidity (ISR for short) property if every G-invariant von Neumann subalgebra $P \subseteq L(G)$ is of the form $P = L(H)$ for some normal subgroup $H \lhd G$.*

Besides the groups considered in [5], two typical groups with the ISR property are the non-abelian free group F_2 [2, 3] and the finitary infinite permutation group S_∞ [4]. For more groups with the ISR property, see [2, 5, 3, 4]. It is also known that being icc is a necessary but not sufficient condition for an infinite group to have the ISR property, see [2, Proposition 3.1 and Example 3.5].

Problem 3.1. *Which infinite conjugate class (ICC) groups G have the ISR property? The conjecture is that:*

- *If G is non-amenable, then G has the ISR property iff G has no amenable normal subgroups other than $\{e\}$.*
- *If G is amenable, then G has the ISR property iff G has no abelian normal subgroups other than $\{e\}$.*

The following question was recorded as [2, Question 4.3]. For partial results, see [2, Theorem 1.4].

Problem 3.2. *If two infinite groups G and H have the ISR property, does $G \times H$ still have the ISR property?*

References

[1] V. Alekseev, R. Brugger, A rigidity result for normalized subfactors, J. Operator Theory, 86, 2021, 3–15.

[2] Amrutam, T. Jiang, Y. On invariant von Neumann subalgebras rigidity property, J. Funct. Anal., 284, 2023,109804, 26

[3] I. Chifan, S. Das, and B. Sun, Invariant subalgebras of von Neumann algebras arising from negatively curved groups, J. Funct. Anal. 285, 2023, 110098.

[4] Y. Jiang, X. Zhou, An example of infinite amenable group with the ISR property, arXiv: 2312.0806, 2023.

[5] M. Kalantar, N. Panagopoulos, On invariant subalgebras of group C^* and von Neumann algebras, Ergodic Theory Dynam. Systems, 43, 2023, 10, 3341–3353.

4. Problems on Dilations and Applications of Operator-valued Measure **(Contributed by Qianfeng Hu and Rui Liu)**

Since 1999, Casazza et al.[4] introduced Banach space Schauder frame, the dilation techniques of frames to base, or more generally, projection valued dilations for operator-valued measure or homomorphism dilation for bounded linear map on operator algebra [10, 9, 7] became a useful and efficient tool to investigate the bounded approximation property (BAP)[12, 15] and its Lipschitz version [5, 16], which supplemented the results by Johnson, Rosenthal, and Zippin [13]. To generalize the fundamental James theorems on base and reflexivity [11] in the general context, Beanland et al.[3] showed that a Schauder frame for any separable Banach space is shrinking if and only if it has an associated space with a shrinking basis, and that a Schauder frame for any separable Banach space is shrinking and boundedly complete if and only if it has a reflexive associated space. It's still open for unconditional (Schauder) frames and bases for reflexive Banach spaces. If, more generally, we consider the operator-valued measure, in particular, the ones induced unconditional frames, then it is natural to consider the duality dilation problem of operator-valued measures on Banach spaces.

Question 1. *Let X be a reflexive Banach space and (Ω, Σ) be a measurable space. Suppose $E : \Sigma \to B(X)$ is an operator-valued measure, is there a reflexive Banach space and projection valued measure F and bounded linear maps $T : X \to Z$, $S : Z \to X$ such that*

$$E(B) = SF(B)T$$

for every $B \in \Sigma$, that is, can every operator-valued measure on a reflexive Banach space be dilated to a projection-valued measure on another reflexive Banach space?

Frame quantum detection problem [6, 2] asks to characterize the informationally completeness of positive operator-valued measure (POVM) induced by Hilbert space (discrete or continuous)frames. We call such frames quantum injective. That is, to find a frame such that frame-induced POVM V with the property that $\mathrm{tr}\,(\rho_1 V(E)) = \mathrm{tr}\,(\rho_2 V(E))$ for all $E \in \Sigma$ implies that $\rho_1 = \rho_2$.

Many interesting or important frames have good structure. The group frames [8, 1] the orbit of a single window vector or function under the unitary representation of a group, have been studied extensively. As the injectivity of the finite abelian group frame has been shown in Li et.al [14], it remains to show the injectivity of other group frames. Through POVMs, the quantum detection problem can be reduced to characterize the properties of window vectors or functions.

Question 2. *Under what conditions of the window vectors or functions of group frames such that the POVMs induced by those group frames are informationally complete? Like, finite non-abelian group: the dihedral group, metacyclic group, or time-frequency representation of $\mathbb{R}^2$ or, the wavelet representation of Affine group $Aff = \mathbb{R} \times \mathbb{R}^*$, etc.*

Recall the frame phase retrieval problem[14]: Given a frame $\{x_j\}_{j\in\mathcal{J}}$, whether the phase-less measurements

$$\{|\langle x, x_j\rangle| : j \in \mathcal{J}\}$$

uniquely determines x (module a scalar)? i.e., $|\langle x, x_j\rangle| = |\langle y, x_j\rangle|$ for every $j \in \mathcal{J}$ implies that $x = \lambda y$ for some $|\lambda| = 1$? Note that the quantum injectivity of frames implies pure state injectivity, which is equivalent to the phase retrieval problem [6].

Question 3. *Given a discrete group G and an irreducible unitary representation π, is there a window vector φ such that the group frame $\{\pi(g)\varphi\}_{g\in G}$ is phase-retrievable but not quantum injective, or it possible that every phase-retrievable group frame gives quantum injectivity.*

References

[1] S.T. Ali, J.P. Antoine and J.P. Gazeau, *Coherent states, wavelets and their generalizations*, New York: Springer, 2000.

[2] S. Botelho-Andrade, P.G. Casazza, D. Cheng and T.T. Tran, *The solution to the frame quantum detection problem*, J. Fourier. Anal. Appl. 25(2019), 2268-2323.

[3] K. Beanland, D. Freeman and R. Liu. *Upper and lower estimates for Schauder frames and atomic decompositions*, Fund. Math. 231(2015), 161-188.

[4] P.G. Casazza, D. Han and D.R. Larson, *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math., Amer. Math. Soc., 247(1999), 149-182.

[5] G. Godefroy and N.J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159(2003), 121-141.

[6] D. Han, Q. Hu and R. Liu. *Injective continuous frames and quantum detections*, Banach J. Math. Anal. 15(2021), 1-24.

[7] D. Han, Q. Hu, D.R. Larson and R. Liu, *Dilations for operator-valued quantum measures*, Adv. Math. 438(2024), 109476.

[8] D. Han and D.R. Larson, *Frames, bases and group representations*, Mem. Am. Math. Soc. Vol. 147. No. 697. 2000.

[9] D. Han and R. Liu, *Frame theory meets operator-valued measures – a dilation point of view*, Proceedings of 2019 ICCM, to appear.

[10] D. Han, D. R. Larson, B. Liu and R. Liu, *Operator-valued measures, dilations and the theory of frames*, Mem. Am. Math. Soc. Vol. 229. No. 1075. 2014.

[11] R.C. James, *Bases and reflexivity of Banach spaces*, Ann. Math. 52(1950), 518–527.

[12] W.B. Johnson and J. Lindenstrauss. *Handbook of the geometry of Banach spaces. Chapter 7: Approximation properties*. Elsevier, Vol 1. 2001.

[13] W.B. Johnson, H.P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. 9(1971), 488–506.

[14] L. Li, T. Juste, J. Brennan, C. Cheng, and D. Han, *Phase retrievable projective representation frames for finite abelian groups*, J. Fourier Anal. Appl. 25(2019), 86-100.

[15] R. Liu and Z. Ruan, *Cb-frames for operator spaces*, J. Funct. Anal. 270(2016), 4280-4296.

[16] R. Liu, J. Shen, and B. Zheng, *Operators with the Lipschitz bounded approximation property*, Sci. China Math. 66(2023), 1545-1554.

5. Problems on Density of Reducible Operators in Factors (**Contributed by Rui Shi**)

In $M_n(\mathbb{C})$, by Jordan canonical form theorem, every $n \times n$ matrix a is similar to a direct sum of Jordan blocks. This means there exists an invertible matrix x in $M_n(\mathbb{C})$ such that

$$x^{-1}ax = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix} \tag{5.1}$$

where each block J_k is a square matrix of the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix} \tag{5.2}$$

Note that no non-trivial Jordan block can be written as a direct sum of two Jordan blocks. In this sense, Jordan blocks can be viewed as fundamental building elements of general matrices. In other words, a Jordan block has no nontrivial reducing subspaces.

Let $\mathcal{H}$ be a nonzero complex separable Hilbert space. Inspired by Jordan matrix in $M_n(\mathbb{C})$, we introduce irreducible operators in $\mathcal{B}(\mathcal{H})$.

Definition 5.1. *An operator a in $\mathcal{B}(\mathcal{H})$ is said to be reducible, if there exists a nontrivial projection p in $\mathcal{B}(\mathcal{H})$ satisfying*

$$pa = ap. \tag{5.3}$$

Otherwise, a is said to be irreducible.

In 1968, Paul Halmos proved in [1] that irreducible operators form a $\|\cdot\|$-dense G_δ subset of $\mathcal{B}(\mathcal{H})$.

In 1970, he proposed the following problem in [2].

Halmos' 8th Problem. *Do reducible operators form a $\|\cdot\|$-dense G_δ subset of $\mathcal{B}(\mathcal{H})$?*

It is clear to check that in $M_n(\mathbb{C})$, reducible operators are nowhere dense. But it is not easy to answer the question on infinite-dimensional separable Hilbert space.

In 1976, Dan Voiculescu proved the non-commutative Weyl-von Neumann theorem in [8]. As its application, he answered Halmos' 8th problem affirmatively by proving that every operator in $\mathcal{B}(\mathcal{H})$ is a $\|\cdot\|$-limit of reducible operators.

In the field of von Neumann algebras, $\mathcal{B}(\mathcal{H})$ is a type I factor. Recall that a $*$-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra if $\mathcal{M}$ contains the identity operator I and closed in the weak-operator topology. Furthermore, a von Neumann algebra $\mathcal{M}$ is said to be a factor if

$$\mathcal{M} \cap \mathcal{M}' \cong \mathbb{C} \tag{5.4}$$

where $\mathcal{M} := \{x \in \mathcal{B}(\mathcal{H}) : xy = yx, \text{ for all } y \in \mathcal{M}\}$.

Francis Murray and John von Neumann developed the fundamental theory of von Neumann algebras in [3, 4, 5, 6, 7] and classified factors in three types. It is also natural to introduce irreducible operators in factors.

Definition 5.2. *Let $\mathcal{M}$ be a factor with separable predual. an operator a in $\mathcal{M}$ is said to be reducible in $\mathcal{M}$ if there exists a nontrivial projection p in $\mathcal{M}$ such that $pa = ap$. Otherwise, a is said to be irreducible in $\mathcal{M}$.*

In 2018, Junsheng Fang, Rui Shi, and Shilin Wen proved in [9] that for every factor $\mathcal{M}$ with separable predual, irreducible operators form a $\|\cdot\|$-dense subset in $\mathcal{M}$.

Thus, we can ask Halmos' 8th problem in factors.

Halmos' 8th Problem (factor version)**.** *Let $\mathcal{M}$ be a factor with separable predual. For every operator a in $\mathcal{M}$, is a can be expressed as a $\|\cdot\|$-limit of reducible operators in $\mathcal{M}$?*

In 2019, Junhao Shen and Rui Shi proved that for a type II_1 factor $\mathcal{M}$ without Property Gamma, reducible operators are nowhere dense in $\mathcal{M}$ in the operator norm. (https://doi.org/10.48550/arXiv.1907.00573) But other cases are still open till now.

References

[1] Paul Halmos. Irreducible operators. Michigan Math J. **15** (1968), 215–223.

[2] Paul Halmos. Ten problems in Hilbert space. Bull. Amer. Math. Soc. **76** (1970), 887–933.

[3] Francis Murray and John von Neumann. On rings of operators. Ann. of Math. (2) **37** (1936), no. 1, 116–229.

[4] Francis Murray and John von Neumann. On rings of operators. II. Trans. Amer. Math. Soc. **41** (1937), no. 2, 208–248.

[5] Francis Murray and John von Neumann. On rings of operators. IV. Ann. of Math. (2) **44** (1943), 716–808.

[6] John von Neumann. On rings of operators. III. Ann. of Math. (2) **41** (1940), 94–161.

[7] John von Neumann. On rings of operators. Reduction theory. Ann. of Math. (2) **50** (1949), 401–485.

[8] Dan Voiculescu. A non-commutative Weyl-von Neumann theorem. Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.

[9] Shilin Wen, Junsheng Fang, Rui Shi. On irreducible operators in factor von Neumann algebras. Linear Algebra Appl. **565** (2019), 239–243.

6. Problems on Quantum Rényi Divergences (**Contributed by Ke Li**)

Rényi's information divergence, defined for two probability densities, is a fundamental information quantity. Its quantum generalization, due to the noncommutativity nature of density operators, can take infinitely many possible forms. To identify the correct quantum generalization of Rényi's information divergence is significant and nontrivial. The key point is that a proper quantum Rényi divergence should have precise operational meaning.

So far, there are two versions of quantum Rényi divergence that admit operational interpretations. One is the sandwiched Rényi divergence [1, 2]

$$D^*_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha. \tag{6.1}$$

The other one is Petz's Rényi divergence [3]

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr}\left(\rho^\alpha\sigma^{1-\alpha}\right). \tag{6.2}$$

In Eq. (6.1) and Eq. (6.2), ρ and σ are density operators, and $\alpha \in (0,1)\cup(1,\infty)$ is a real parameter. Operational interpretations of D^*_α with $\alpha \geqslant \frac{1}{2}$ are obtained, e.g., in Refs [4, 5, 6], and those of D_α with $\alpha\in(0,1)$ can be seen in [7].

Problem 6.1. *Can we find an operational interpretation for D_α with $\alpha \in (1,2)$? D_α has nice properties in this interval, and the duality relation of* [8] *indicates that the case $\alpha\in(1,2)$ is special.*

Problem 6.2. *What is the full picture of the quantum generalization of Rényi's information divergence? Is there any quantum Rényi divergence, other than D_α and D^*_α, admitting operationally meaning?*

Problem 6.3. *For a bipartite density operator ρ_{AB} on Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$, define*

$$I^*_\alpha(A:B)_\rho := \min_{\sigma_A,\sigma_B} D^*_\alpha(\rho_{AB}\|\sigma_A\otimes\sigma_B),$$

where the minimization is over all density operators σ_A on $\mathcal{H}_A$ and σ_B on $\mathcal{H}_B$. We ask whether I^*_α is additive, in the sense that

$$I^*_\alpha(A_1A_2:B_1B_2)_{\rho\otimes\omega} = I^*_\alpha(A_1:B_1)_\rho + I^*_\alpha(A_2:B_2)_\omega$$

for any density operators $\rho_{A_1B_1}$ and $\omega_{A_2B_2}$, and for all $\alpha\geqslant\frac{1}{2}$. This additivity property was shown in the classical case for $\alpha\in[\frac{1}{2},\infty)$ [9]*, and recently proved in the quantum case for $\alpha\in(1,\infty)$* [10]*. It was conjectured in* [6] *that it holds as well for $\alpha\in[\frac{1}{2},1)$. However, the methods of* [9, 10] *do not seem to work here.*

References

[1] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel. On quantum Rényi entropies: a new generalization and some properties. J. Math. Phys., 54:122203, 2013.

[2] M. M. Wilde, A. Winter, and D. Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. Commun. Math. Phys., 331(2):593–622, 2014.

[3] D. Petz. Quasi-entropies for finite quantum systems. Rep. Math. Phys., 23(1):57–65, 1986.

[4] M. Mosonyi and T. Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. Commun. Math. Phys., 334(3):1617–1648, 2015.

[5] K. Li and Y. Yao. Reliability function of quantum information decoupling via the sandwiched Rényi divergence. arXiv:2111.06343, 2021.

[6] K. Li and Y. Yao. Operational interpretation of the sandwiched Rényi divergence of order 1/2 to 1 as strong converse exponents. Commun. Math. Phys., 405:22, 2024.

[7] K. Audenaert, M. Nussbaum, A. Szkoła, and F. Verstraete. Asymptotic error rates in quantum hypothesis testing. Commun. Math. Phys., 279(1):251–283, 2008.

[8] M. Hayashi and M. Tomamichel. Correlation detection and an operational interpretation of the Rényi mutual information. J. Math. Phys., 57:102201, 2016.

[9] M. Tomamichel and M. Hayashi. Operational interpretation of Rényi information measures via composite hypothesis testing against product and Markov distributions. IEEE Trans. Inf. Theory, 64(2):1064–1082, 2017.

[10] H.-C. Cheng and L. Gao. Tight one-shot analysis for convex splitting with applications in quantum information theory. arXiv:2304.12055, 2023.

7. Problems on the parafermion vertex operator algebras (**Contributed by Qing Wang**)

Coset construction and orbifold construction are two basic ways to construct new vertex operator algebras from given ones. Parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is a special kind of coset construction related to affine vertex operator (super)algebras. It is the commutant of Heisenberg vertex operator algebra in the simple affine vertex operator (super)algebra $L_{\hat{\mathfrak{g}}}(k, 0)$, where $L_{\hat{\mathfrak{g}}}(k, 0)$ is the integrable highest weight module with the positive integer level k for affine Lie algebra $\hat{\mathfrak{g}}$ associated to the basic classical simple Lie superalgebra $\mathfrak{g}$. The structure and representation theory of $K(\mathfrak{g}, k)$ has been fully studied in the past ten years (see [1, 2, 3, 4, 5, 6, 7, 9, 10] ect.). The natural problem next is the orbifold theory of the parafermion vertex operator algebra. From the generator results of the parafermion vertex operator algebras associated to any basic classical simple Lie superalgebras[8], we know that the parafermion vertex operator algebras associated to sl_2 and $osp(1|2)$ are the building blocks for the rational parafermion vertex operator algebras. So it is important to first understand the representation theory and orbifold theory of the parafermion vertex operator algebras associated to sl_2 and $osp(1|2)$.

Problem 7.1. *The classification of the irreducible modules of the rational parafermion vertex operator algebra $K(osp(1|2n), k)$ and fusion rules.*

Problem 7.2. *The orbifold theory of the rational parafermion vertex operator algebra $K(osp(1|2n), k)$ and their fusion rules.*

Problem 7.3. *The structure and representation theory of $K(osp(1|2n), k)$ at critical level.*

References

[1] C. Ai, C. Dong, X. Jiao and L. Ren, The irreducible modules and fusion rules for the parafermion vertex operator algebras. Trans. Amer. Math. Soc. **370**(2018), 5963-5981.

[2] T. Arakawa, C.H. Lam and H. Yamada, Zhu's algebra, C_2-cofiniteness of parafermion vertex operator algebras, Adv. Math. **264** (2014), 261-295.

[3] T. Arakawa, C.H. Lam and H. Yamada, Parafermion vertex operator algebras and W-algebras, Trans. Amer. Math. Soc. **371**(2019), 4277-4301.

[4] C. Dong, C.H. Lam and H. Yamada, W-algebras related to parafermion algebras, J. Algebra **322** (2009), 2366-2403.

[5] C. Dong and Q. Wang, The structure of parafermion vertex operator algebras: general case, Comm. Math. Phys. **299** (2010), 783-792.

[6] C. Dong and Q. Wang, On C_2-cofiniteness of parafermion vertex operator algebras, J. Algebra **328** (2011), 420-431.

[7] C. Dong and Q. Wang, Quantum dimensions and fusion rules for parafermion vertex operator algebras, Proc. Amer. Math. Soc. **144** (2016), 1483-1492.

[8] C. Jiang, A.R. Linshaw, Q. Wang, The structure of parafermion vertex operator algebras: general case, in preparation.

[9] C. Jiang and Q. Wang, The structure of parafermion vertex operator algebras $K(osp(1|2n), k)$, J. Algebra **594** (2022), 722-732.

[10] Q. Wang, Automorphism group of parafermion vertex operator algebras, J. Pure Appl. Alg. **220** (2016), 94-107.

8. How to derive three-point function for critical ($\gamma = 2$) Liouville CFT? (**Contributed by Baojun Wu**)

The Liouville conformal field theory (Liouville CFT) was introduced by A. Polyakov in his path integral formulation of String Theory [Pol81] and it served as a motivation for Belavin, Polyakov, and Zamolodchickov in their work on conformal field theory [BPZ84]. It also plays a fundamental role in the study of random surfaces, statistical physics, $4d$ supersymmetric Yang-Mills theory, and many other fields of physics and mathematics. It corresponds to taking the particular action functional, called Liouville action, defined for $\phi : \Sigma \to \mathbb{R}$ on closed Riemann surface (Σ, g) by

$$S_\Sigma(g,\phi) := \frac{1}{4\pi}\int_\Sigma \left(|d\phi|_g^2 + QK_g\phi + 4\pi\mu e^{\gamma\phi}\right) \mathrm{dv}_g. \tag{8.1}$$

where K_g is the scalar curvature and v_g the volume form on Σ determined by the metric g. The parameters of LCFT are cosmology constant $\mu > 0$, $\gamma \in (0,2)$ and background charge $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. The LCFT is described in terms of positive measure on a set $\mathcal{D}(\Sigma)$ of (generalized) real-valued functions ϕ on Σ. Expectation (denoted by $\langle\cdot\rangle_g^\Sigma$ in what follows) under this measure is formally given as a *path integral*

$$\langle F\rangle_g^\Sigma = \int_{\mathcal{D}(\Sigma)} F(\phi)e^{-S_\Sigma(g,\phi)}D\phi \tag{8.2}$$

for suitable observables $F : \mathcal{D}(\Sigma) \to \mathbb{C}$ and $D\phi$ a formal Lebesgue measure on $\mathcal{D}(\Sigma)$. The basic observables in LCFT are the vertex operators, which are formally defined by $e^{\alpha\phi(z)}$, $\alpha \in \mathbb{C}$ and $z \in \Sigma$. The fundamental objects in the LCFT are correlation functions, which can be described by path integral as follows. For $z_1, z_2, ..., z_n \in \Sigma$ and $\alpha_1, \alpha_2, .., \alpha_n \in \mathbb{C}$

$$\langle\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle_g^\Sigma \tag{8.3}$$

This theory, with central charge $c_{\mathrm{L}} := 1 + 6Q^2$, has been extensively studied in theoretical physics. This theory is constructed for $\gamma \leqslant 2$ case in [DKRV16]. The fundamental problem in Liouville CFT is to compute the three-point function on the Riemann sphere explicitly, this is achieved for $\gamma < 2$ case in [KRV17].

In the $\gamma = 2$ case (also called the critical case), the mathematical definition of Liouville correlation function needs a different renormalization from the $\gamma < 2$ case. The three-point function in the $\gamma = 2$ case is still finite, see [DKRV16, section 4]. One can ask how to compute the three-point function explicitly in this case.

One possibility to solve this is using the strategy from Liouville quantum gravity and statistical physics. In [AS21], Ang and Sun derive the joint law of CLE_{γ^2} conformal radius by combining the mating of trees strategy from [DMS14] and Liouville CFT. They first show the $\gamma < 2$ case, then use the coupling between CLE and Brownian loop soups to analysis the $\gamma \to 2$ behavior. The coupling between CLE_4 and $\gamma = 2$ Liouville theory is also developed recently in [AG23]. Can we use the joint law of CLE_4 conformal radius CLE_4 to compute $\gamma = 2$ three-point function in Liouville CFT?

References

[AG23] Morris Ang and Ewain Gwynne. Critical liouville quantum gravity and cle$_4$, ArXiv:2308.11835, 2023.

[AS21] Morris Ang and Xin Sun. Integrability of the conformal loop ensemble, ArXiv:2107.01788 , 2021.

[BPZ84] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. Nucl. Phys., B241:333–380, 1984. [,605(1984)].

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville Quantum Gravity on the Riemann sphere. Commun. Math. Phys., 342(3):869–907, 2016.

[DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. ArXiv:1409.7055, September 2014.

[KRV17] A. Kupiainen, R. Rhodes, and V. Vargas. Integrability of Liouville theory: proof of the DOZZ Formula. Annals of Mathematics, to appear, 2017.

[Pol81] A. M. Polyakov. Quantum geometry of bosonic strings. Phys. Lett. B, 103(3):207–210, 1981.

9. Problems on Tensor Category arising from Affine Vertex Operator Algebras (Contributed by Jinwei Yang)

Tensor category structures on the representation category of affine Lie algebra $\widehat{\mathfrak{g}}$ have been studied extensively by physicists and mathematicians since late 1980s. Mathematically, D. Kazhdan and G. Lusztig ([KL1]-[KL5]) first constructed braided tensor category on the category of finite length modules whose composition factors are integrable simple $\widehat{\mathfrak{g}}$-modules when the level k plus the dual Coexter number $h^\vee$ is not positive rational, they also showed that this category is braided tensor equivalent to the category of finite dimensional weight modules for the quantum group $U_q(\mathfrak{g})$ for $q = e^{\frac{\pi i}{r^\vee(k+h^\vee)}}$, where $r^\vee$ is the lacety of the finite dimensional Lie algebra $\mathfrak{g}$. Consequently, this category of affine Lie algebra modules is rigid.

When the level k is positive integral, Y.-Z. Huang ([H1]) proved the category of $\widehat{\mathfrak{g}}$-modules of level k that are isomorphic to direct sums of integrable simple $\widehat{\mathfrak{g}}$-modules of level k is a modular tensor category, using (logarithmic) tensor category theory of vertex operator algebras developed by Huang, J. Lepowsky and L. Zhang.

When the level k is admissible, i.e., $k + h^\vee = \frac{p}{q}$ with $(p, q) = 1$, $p, q \in \mathbb{Z}_{\geqslant 1}$ and

$$p \geqslant \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where h is the Coexter number of $\mathfrak{g}$, T. Creutzig, Huang and J. Yang ([CHY]) constructed braided tensor categories on the category $KL_k(\mathfrak{g})$ of $\widehat{\mathfrak{g}}$-modules of level k that are isomorphic to direct sums of simple modules for the simple affine vertex operator algebra $L_k(\mathfrak{g})$. Later on, Creutzig ([C]) showed that these categories are rigid if $\mathfrak{g}$ is of type ADE and Creutzig, N. Genra and A. Linshaw ([CGL]) proved that the category is rigid if $\mathfrak{g}$ is of type C. Rigidity for the remaining cases is still open:

Conjecture 9.1. *The category $KL_k(\mathfrak{g})$ is rigid for all $\mathfrak{g}$, and thus is a ribbon tensor category.*

It is believed that there are correspondences, usually called Kazhdan-Lusztig correspondence, between $KL_k(\mathfrak{g})$ and the braided tensor categories arising from the quantum groups:

Conjecture 9.2. *The category $KL_k(\mathfrak{g})$ is braided equivalent to the semisimplification of category of tilting modules for the quantum group $U_q(\mathfrak{g})$ for $q = e^{\frac{\pi i}{r^\vee(k+h^\vee)}}$.*

Combining with the modularity of the tensor category arising from quantum groups (see for example [R]), Conjecture 9.2 also implies:

Conjecture 9.3. *The category $KL_k(\mathfrak{g})$ (admissible $k = -h^\vee + \frac{p}{q}$) are modular in the following cases:*

(1). $\mathfrak{g} = \mathfrak{sl}_n$ *if and only if* $(q, n) = 1$.

(2). $\mathfrak{g} \in \{\mathfrak{so}_{4n+1}, \mathfrak{sp}_{2n}\}$ *if and only if* q *odd.*

(3). $\mathfrak{g} = \mathfrak{so}_{4n+3}$ *if and only if* q *odd or* $q/2$ *odd.*

(4). $\mathfrak{g} \in \{\mathfrak{so}_{4n}, \mathfrak{e}_7, \mathfrak{f}_4\}$ *if* q *odd.*
(5). $\mathfrak{g} = \mathfrak{so}_{4n+2}$ *if* $(q, 4) = 1$.
(6). $\mathfrak{g} = \{\mathfrak{e}_6, \mathfrak{g}_2\}$ *if* $(q, 3) = 1$.
(7). $\mathfrak{g} = \mathfrak{e}_8$ *for all* q.

When the level plus the dual Coexter number is positive rational, but not admissible, we also would like to know:

Conjecture 9.4. *What tensor category structures can we construct? Are they rigid, semisimple or modular?*

There are some progress on Conjecture 9.4 in [CY], but in general the conjecture is still widely open. For more details related to this problem set, please see also [CHY] and [H2].

References

[C] T. Creutzig, Fusion categories for affine vertex algebras at admissible levels, Selecta Math. (N.S.), **25(2)**: Paper No. 27, 21, 2019.

[CGL] T. Creutzig, N. Genra and A. Linshaw, Category $\mathcal{O}$ for vertex algebras of $\mathfrak{osp}_{1|2n}$, arXiv:2203.08188.

[CHY] T. Creutzig, Y.-Z. Huang and J. Yang, Braided tensor categories of admissible modules for affine Lie algebras, Comm. Math. Phys. **362** (2018), 827–854.

[CY] T. Creutzig and J. Yang, Tensor categories of affine Lie algebras beyond admissible levels, Math. Ann. **380** (2021), no. 3-4, 1991–2040.

[H1] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, Comm. Contemp. Math. **10** (2008), 871–911.

[H2] Affine Lie algebras and tensor categories, 14 pages, Proceedings of 10th CFT Seminar: A Conference on Vertex Algebras and Related Topics at RIMS, to appear.

[KL1] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, Duke Math. J., IMRN **2** (1991), 21–29.

[KL2] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I, J. Amer. Math. Soc. **6** (1993), 905–947.

[KL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, II, J. Amer. Math. Soc. **6** (1993), 949–1011.

[KL4] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, III, J. Amer. Math. Soc. **7** (1994), 335–381.

[KL5] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, IV, J. Amer. Math. Soc. **7** (1994), 383–453.

[R] E.C. Rowell, From quantum groups to unitary modular tensor categories. In: Representations of Algebraic Groups, Quantum Groups, and Lie Algebras, volume 413 of Contemp. Math., pp. 215–230. Amer. Math. Soc., Providence, RI (2006).

10. A Problem on L_1 Poincare Inequality (**Contributed by Qiang Zeng**)

Let (N,τ) be a noncommutative W^* probability space, where N is a finite von Neumann algebra and τ is a normal faithful tracial state. Let $P_t, t \geqslant 0$ be a noncommutative symmetric Markov semigroup (i.e., each P_t is unital completely positive and trace preserving) acting on (N,τ) with generator $-A$ on $L_2(N,\tau)$, where A is positive on $L_2(N,\tau)$. We may define Meyer's carré du champ on a dense subset of N:

$$\Gamma(f_1,f_2) = \frac{1}{2}[A(f_1^*)f_2 + f_1^*A(f_2) - A(f_1^*f_2)]$$

and

$$\Gamma_2(f_1,f_2) = \frac{1}{2}[\Gamma(Af_1^*,f_2) + \Gamma(f_1^*,A(f_2)) - A\Gamma(f_1^*f_2)],$$

whenever these quantities are well-defined for $f_1,f_2 \in N$. Assume the fixed point algebra of P_t is trivial so that $\lim_{t\to\infty}\|P_tf - \tau(f)\|_2 = 0$ for $f \in L_2(N,\tau)$. A classic example of this setting is the Ornstein–Uhlenbeck semigroup on $N = L_\infty(\mathbb{R}^d,\gamma_d)$, where γ_d is the d-dimensional standard Gaussian measure on the Euclidean space $\mathbb{R}^d$. In this case, we have $-A = \Delta - x\cdot\nabla$, $\Gamma(f,f) = |\nabla f|^2$ and

$$\Gamma_2(f,f) = |\nabla f|^2 + \|\nabla^2 f\|_2,$$

where $\|\nabla^2 f\|_2$ is the Hilbert–Schmidt norm of the Hessian of f. In this setting, we have $\Gamma(f,f) \leqslant \Gamma_2(f,f)$. This condition can be extended to much more general setting and is commonly known as the Bakry–Emery condition $\Gamma_2(f,f) \geqslant \alpha\Gamma(f,f)$ for $\alpha \in \mathbb{R}$. On a Riemannian manifold, this condition is equivalent to that the Ricci curvature is bounded from below by α (see e.g. [Le04, Le11]).

In the general noncommutative setting, the following problem is open:

Problem 10.1. *Suppose $\Gamma_2(f,f) \geqslant 0$ and $\|f-\tau(f)\|_2 \leqslant C\|\Gamma(f,f)^{1/2}\|_2$ (i.e., spectral gap exists) for some absolute constant $C>0$. Prove that*

$$\|f-\tau(f)\|_1 \leqslant C'\|\Gamma(f,f)^{1/2}\|_1$$

for some $C'>0$.

In classical probability and analysis, the conclusion holds. Indeed, we consider a metric measure space (X,d,μ) equipped with a separable Borel probability measure μ. Let B be a Borel set of X and define the surface measure of B as

$$\mu^+(B) = \liminf_{\varepsilon\to 0+}\frac{\mu(B^\varepsilon)-\mu(B)}{\varepsilon}$$

where $B^\varepsilon = \{x\in X : \exists a \in B, d(x,a) < \varepsilon\}$ is the open ε-neighborhood of B. The modulus of the gradient of a function f on X is

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0+}\frac{|f(x)-f(y)|}{d(x,y)}$$

and assign $|\nabla f(x)| = 0$ if x is an isolated point. In this context, Cheeger's inequality states that there exists a constant $c > 0$ such that

$$\mu(A^+) \geqslant c \min\{\mu(A), \mu(A^c)\}, \quad A \subset X \text{ Borel.}$$

This is a type of isoperimetric inequalities, which compare the surface area of the boundary of a set with its volume. It is well known that Cheeger's inequality is equivalent to the following L_1 Poinaré inequality (see e.g. [BH97, Theorem 1.1] and also [Le11]):

$$\int_X |\nabla f| \mathrm{d}\mu \geqslant c \left\| f - \int_X f \mathrm{d}\mu \right\|_{L_1(\mu)}.$$

A classic result of Cheeger showed that Cheeger's inequality implies the existence of spectrum gap, or equivalently the ordinary L_2 Poincaré inequality

$$\left\| f - \int_X f \mathrm{d}\mu \right\|_{L_2(\mu)}^2 \leqslant C^2 \int_X |\nabla f|^2 \mathrm{d}\mu;$$

see e.g. [Le04]. The relationship among the three inequalities can be presented as

$$L_1 \text{ Poincaré inequality} \Leftrightarrow \text{Cheeger's inequality} \Rightarrow \text{Spectral gap exists.}$$

The proposed problem is about the reversed direction of the implication under additional conditions. In [Le04, Theorem 5.2], Ledoux showed that in the Riemannian setting, the spectral gap and a lower bound on Ricci curvature imply Cheeger's inequality. This result goes back to Buser, and there are also discrete versions; see [Le04] for more details.

The above discussion shows that the proposed problem is known in the classical setting. Part of the proof is based on semigroups, which is relatively simple to extend to the noncommutative setting. On the other hand, some properties of the underlying space are used in a crucial way in the proof. To quote Ledoux [Le11], "In view of the preceding semigroup argument, such an additional ingredient seems indeed unavoidable ..." Since there is no underlying space in the general noncommutative setting, the proposed problem cannot be solved using the known commutative theory. For instance, in the noncommutative setting, it is not even known how to formulate Cheeger's inequality so that it is equivalent to the L_1 Poincaré inequality.

References

[BH97] S. Bobkov, C. Houdré, Some connections between isoperimetric and Sobolev-type inequalities, Mem. Amer. Math. Soc., vol. 129, 1997.

[Le04] M. Ledoux, Spectral gap, logarithmic Sobolev constant, and geometric bounds, Surveys in Differential Geometry IX, International Press, 2004.

[Le11] M. Ledoux, From Concentration to Isoperimetry: Semigroup Proofs, 2011.

11. Problems in Vertex Operator Algebras (**Contributed by Hao Zhang and Bin Gui**)

Let $\mathbb{V} = \oplus_{n\in\mathbb{N}}\mathbb{V}(n)$ be a C_2-cofinite vertex operator algebra. We defined a $\mathbb{V}^{\otimes M}$-module $\boxbslash_{\mathfrak{X}}(\mathbb{W})$, called the *dual fusion product* associated to a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ and an (M,N)-pointed compact Riemann surface

$$\mathfrak{X} = (y_1,\cdots,y_M|C|x_1,\cdots,x_N),$$

where $y_\bullet$ (resp. $x_\bullet$) are outgoing (resp. incoming) marked points [GZ23]. The contragredient module of $\boxbslash_{\mathfrak{X}}(\mathbb{W})$ is called the *fusion product* and is denoted by $\boxtimes_{\mathfrak{X}}(\mathbb{W})$. When $\mathbb{W}$ is a tensor product of two $\mathbb{V}$-modules and $\mathfrak{X} = (\infty|\mathbb{P}^1|0,1)$, $\boxtimes_{\mathfrak{X}}(\mathbb{W})$ concides with the fusion product in the braided tensor category $\mathrm{Rep}(\mathbb{V})$ defined in [HLZ14, HLZ12a, HLZ12b, HLZ12c, HLZ12d, HLZ12e, HLZ12f, HLZ12g].

One of our final goal is to prove that $\mathrm{Rep}(\mathbb{V})$ is a modular tensor category. It remains to prove:

Problem 11.1. *Assume that $\mathbb{V}$ is self-dual.* $\mathrm{Rep}(\mathbb{V})$ *is actually rigid (and hence modular).*

When $\mathbb{V}$ is also rational, the proof of Problem 11.1 reduces to proving that some fusion coefficients are nonzero. Geometrically, this kind of fusion coefficients is closely related to modular invariance property [Zhu96, Hua05], or equivalently, sewing and factorization of genus 1 and genus 0 conformal blocks.

In my talk in Sanya, with the help of dual fusion products, I gave a new version of sewing and factorization theorem when $\mathbb{V}$ is not assumed to be rational. It is a higher genus version but when focusing on genus 1 and genus 0, we have the following factorization isomorphism related to genus 1 and 1 point conformal blocks

$$\mathscr{T}^*_{\mathfrak{T}}(\mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\boxtimes_{\mathbb{P}^1}(\mathbb{V}), \boxbslash_{\mathbb{P}^1}(\mathbb{W})\big). \tag{11.1}$$

Problem 11.2. *What is the relationship between* (11.1) *and the modular invariance given in* [Miy04, Hua23]*? More precisely, what is the relationship between Miyamoto's pseudotraces and dual fusion products?*

Algebraic geometrists are interested in the vector bundle structure of conformal blocks. In [DGT21, DGT22], the authors proved that: when $\mathbb{V}$ is C_2-cofinite and rational, conformal blocks will give vector bundle structures on moduli space of stable pointed curves and satisfy a higher genus factorization property. It is natural to ask:

Problem 11.3. *When $\mathbb{V}$ is only C_2-cofinite, is it possible that conformal blocks give vector bundle structures on moduli of stable pointed curves?*

I believe the answer will be negative. [DGK23] proposed a condition called *smoothing* to solve this problem. But unfortunately, no examples of C_2-cofinite and irrational vertex operator algebras are known to satisfy the smoothing condition.

References

[DGK23] Chiara Damiolini, Angela Gibney, and Daniel Krashen. Conformal blocks on smoothings via mode transition algebras. arXiv:2307.03767, 2023.

[DGT21] Chiara Damiolini, Angela Gibney, and Nicola Tarasca. Conformal blocks from vertex algebras and their connections on $\overline{\mathcal{M}}_{g,n}$. Geom. Topol., 25(5):2235–2286, 2021.

[DGT22] Chiara Damiolini, Angela Gibney, and Nicola Tarasca. On factorization and vector bundles of conformal blocks from vertex algebras. Ann. Sci. Éc. Norm. Supér., 2022.

[GZ23] Bin Gui and Hao Zhang. Analytic conformal blocks of C_2-cofinite vertex operator algebras I: Propagation and dual fusion products. arXiv:2305.10180, 2023.

[HLZ12a] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, ii: Logarithmic formal calculus and properties of logarithmic intertwining operators. arXiv:1012.4196, 2012.

[HLZ12b] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, iii: Intertwining maps and tensor product bifunctors. arXiv:1012.4197, 2012.

[HLZ12c] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, iv: Constructions of tensor product bifunctors and the compatibility conditions. arXiv:1012.4198, 2012.

[HLZ12d] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, v: Convergence condition for intertwining maps and the corresponding compatibility condition. arXiv:1012.4199, 2012.

[HLZ12e] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, vi: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms. arXiv:1012.4202, 2012.

[HLZ12f] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, vii: Convergence and extension properties and applications to expansion for intertwining maps. arXiv:1110.1929, 2012.

[HLZ12g] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory, viii: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra. arXiv:1110.1931, 2012.

[HLZ14] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: introduction and strongly graded algebras and their generalized modules. In Conformal field theories and tensor categories, Math. Lect. Peking Univ., pages 169–248. Springer, Heidelberg, 2014.

[Hua05] Yi-Zhi Huang. Differential equations, duality and modular invariance. Commun. Contemp. Math., 7(5):649–706, 2005.

[Hua23] Yi-Zhi Huang. Modular invariance of (logarithmic) intertwining operators, 2023.

[Miy04] Masahiko Miyamoto. Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. Duke Math. J., 122(1):51–91, 2004.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc., 9(1):237–302, 1996.

12. Problems on Bohnenblust-Hille Inequalitiy (**Contributed by Haonan Zhang**)

In 1931, Bohnenblust and Hille [2] proved a remarkable inequality for the norm of Fourier transform on polytorus. This inequality now is called Bohnenblust-Hille (BH) inequality. In 2001, Blei [1] proved the BH inequality for Boolean cubes, and recently it was reproved by Defant, Mastyło, and Pérez [3] with a much better constant.

Suppose f is a complex-valued function on the Boolean cubes $\{-1,1\}^n$. The Fourier expansion of f is defined as

$$f = \sum_{S \subseteq \{1,2\ldots,n\}} \hat{f}(S)\chi_S,$$

where χ_S is the characteristic function defined as

$$\chi_S(x) := \prod_{j \in S} x_j, \quad x = (x_j)_{j=1}^n.$$

The function f is said to have degree-d if $\hat{f}(S) = 0$ whenever $|S| > d$, where $|S|$ is the cardinality of S. This kind of function has low complexity in the learning theory.

The BH inequality states that for a fixed $d > 1$ and any function f with degree $\leqslant d$, there exists $C_d > 0$ independent of n,

$$\|\hat{f}\|_{\frac{2d}{d+1}} := \left(\sum_{|S| \leqslant d} |\hat{f}(S)|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}} \leqslant C_d \|f\|_\infty.$$

The best constant $\mathbf{BH}^{\leqslant d}_{\{\pm 1\}}$ of the BH inequality satisfies $\mathbf{BH}^{\leqslant d}_{\{\pm 1\}} \leqslant C^{\sqrt{d \log d}}$ for some universal $C > 0$ [3]. This BH inequality for $\{-1,1\}^n$ plays an important role in learning functions $f : \{-1,1\}^n \to [-1,1]$ of low degree using $\mathcal{O}(\log n)$ random queries [4]. In fact, the sample complexity has an explicit upper bound in terms of $\mathbf{BH}^{\leqslant d}_{\{\pm 1\}}$. Thus it is important to improve the bound of the best constant $\mathbf{BH}^{\leqslant d}_{\{\pm 1\}}$.

It is extremely interesting to study the BH inequality for a larger category of symmetries such as cyclic groups, more general discrete groups, and quantum symmetries. In 2023, the BH inequality for qubits was established by Huang-Chen-Preskill [5] and Volberg-Zhang [6], which can be regarded as a noncommutative version of BH inequality.

Problem 12.1. *What is the best constant of the Bohnenblust-Hille inequalities for known symmetries? The quantum torus is an analogy of torus in noncomutative geometry (we refer to* [7] *for the details of quantum tori and reference therein). Does Bohnenblust-Hille inequality hold for quantum tori?*

- The BH inequality could be formulated for planar algebras. We guess that the best constant should depend on the depth of the planar algebra. However, the difficulty is to find a nice sufficient condition for the inequality. (Jinsong Wu)

References

[1] Ron Blei, Analysis in integer and fractional dimensions, volume 71. Cambridge University Press, 2001.

[2] H. F. Bohnenblust and Einar Hille, On the absolute convergence of Dirichlet series. Annals of Mathematics, 32(3):600–622, 1931.

[3] Andreas Defant, Mieczyslaw Mastylo, and Antonio Pérez. On the Fourier spectrum of functions on boolean cubes. Mathematische Annalen, 374(1-2):653–680, September 2018.

[4] Alexandros Eskenazis and Paata Ivanisvili. Learning low-degree functions from a logarithmic number of random queries. In Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, pages 203–207, 2022.

[5] Hsin-Yuan Huang, Sitan Chen, and John Preskill. Learning to predict arbitrary quantum processes. PRX Quantum, 4(4):040337, 2023.

[6] Alexander Volberg and Haonan Zhang. Noncommutative Bohnenblust–Hille inequalities. Mathematische Annalen, page 1–20, 2023.

[7] Xiao Xiong, Quanhua Xu, and Zhi Yin, Sobolev, Besov and Triebel-Lizorkin spaces on quantum tori, Memoirs of the American Mathematical Society, vol 252, 2018,

13. Problems on Quasi-local Algebras (**Contributed by Jiawen Zhang**)

Given a discrete metric space (X, d) of bounded geometry, we can associate the uniform Roe algebra $C^*_u(X)$ and the uniform quasi-local algebra $C^*_{uq}(X)$. These are C^*-subalgebras in $\mathfrak{B}(\ell^2(X))$, coming from higher index theory [3, 4]. It was proved in [4] that if the underlying space X has Yu's Property A [5], then $C^*_u(X) = C^*_{uq}(X)$. Recently, it was proved by Ozawa in [2] that if the space X contains a sequence of expander graphs, then $C^*_u(X) \neq C^*_{uq}(X)$. However, the general picture is far from clear. We are interested in the following question:

Problem 13.1. *For a discrete metric space (X, d) of bounded geometry, can we characterise $C^*_u(X) = C^*_{uq}(X)$ using the coarse geometry of the underlying space? More precisely, does $C^*_u(X) = C^*_{uq}(X)$ implies that X has Yu's Property A?*

We are also interested in their K-theories. Hence we also ask the following:

Problem 13.2. *Can we have a criterion to ensure that $C^*_u(X)$ and $C^*_{uq}(X)$ have the same K-theories (e.g., X can be coarsely embedded into Hilbert space)?*

Furthermore, we can consider the more general groupoid setting [1] and ask similar questions:

Problem 13.3. *For a locally compact Hausdorff and étale groupoid, can we characterise $C^*_r(\mathcal{G}) = C^*_{uq}(\mathcal{G})^{\mathcal{G}}$ using certain property of $\mathcal{G}$? More precisely, does $C^*_u(X) = C^*_{uq}(X)$ implies that $\mathcal{G}$ is (topologically) amenable?*

References

[1] Baojie Jiang, Jiawen Zhang, and Jianguo Zhang. Quasi-locality for étale groupoids. Communications in Mathematical Physics, pages 1–51, 2023.

[2] Narutaka Ozawa. Embeddings of matrix algebras into uniform roe algebras and quasi-local algebras. arXiv preprint arXiv:2310.03677, 2023.

[3] John Roe. An index theorem on open manifolds. I, II. J. Differential Geom., 27(1):87–113, 115–136, 1988.

[4] Ján Špakula and Jiawen Zhang. Quasi-locality and Property A. J. Funct. Anal., 278(1):108299, 25, 2020.

[5] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math., 139(1):201–240, 2000.

14. Problems on Irreducible Approximation (**Contributed by Sen Zhu**)

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the collection of bounded linear operators on $\mathcal{H}$. An *invariant subspace* of an operator $T \in \mathcal{B}(\mathcal{H})$ means a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T(\mathcal{M}) \subset \mathcal{M}$. We let Lat T denote the collection of invariant subspaces of T. Invariant subspaces are suitable infinite-dimensional substitutes of eigenvalues for matrices. Usually, it is difficult to describe Lat T. A fundamental problem in operator theory is the so called *Invariant Subspace Problem*, which asks whether every bounded linear operator T on an separable infinite dimensional Hilbert space has a nontrivial invariant subspace [5]. The problem remains open till now.

Naturally, people turn to consider a special subclass of Lat T, that is, $\mathrm{Red}T \triangleq \mathrm{Lat}\ T \cap \mathrm{Lat}\ T^*$. Here T^* denotes the adjoint of T. Each element of $\mathrm{Red}T$ is called a *reducing subspace* of T. Clearly, $\{0\}, \mathcal{H} \in \mathrm{Red}T$. T is said to be *irreducible* if $\mathrm{Red}T = \{\{0\}, \mathcal{H}\}$; otherwise, T is said to be *reducible*. Irreducible operators have the simplest lattices of reducing subspaces, and can be viewed as the smallest operator units in the reduction sense. So, in order to understand a special class of operators, it is basic to classify those irreducible ones in it. Still, it is often difficult to achieve this.

In 1968, Halmos [3] proved an interesting approximation result, which provides an approximation approach to the study of reducing subspaces.

Theorem 14.1. *The set of irreducible operators on a separable Hilbert space is a dense G_δ set.*

The preceding result shows that irreducible operators on $\mathcal{H}$ constitute a topologically large subset of $\mathcal{B}(\mathcal{H})$. In 1970, P. Halmos [4] raised ten problems in Hilbert spaces and his Problem 8 asked: *Is every operator the norm limit of reducible ones?* In 1976, D. Voiculescu [11] proved the well-known noncommutative Weyl-von Neumann theorem, solving Halmos' problem in the positive.

These results inspire the following terminology. A subset $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ is said to have the *irreducible (reducible) approximation property*, if those irreducible (reducible) ones in $\mathcal{F}$ are norm dense in $\mathcal{F}$. For convenience, we write IAP for "irreducible approximation property" and write RAP for "reducible approximation property". Hence the results of P. Halmos and D. Voiculescu show that $\mathcal{B}(\mathcal{H})$ simultaneously have the IAP and the RAP.

To understand the structure of a special operator class $\mathcal{F} \subset \mathcal{B}(\mathcal{H})$, it is natural to consider whether $\mathcal{F}$ enjoys the IAP or the RAP. In the finite-dimensional case, it is easy to see that any set $\mathcal{F}$ can not have the IAP and RAP simultaneously. Also one can show that many classes of matrices have the IAP such as the classes of Toeplitz matrices, Hankel matrices, stochastic matrices, complex symmetric matrices, and skew-symmetric matrices of order greater than 2 (see [6]).

In the infinite dimensional case, several progresses have been made. By a result of D. Herrero and C. Jiang [7], the class of operators with connected spectra enjoy the IAP. By a result of Y. Ji and C. Jiang [8], the class of Cowen-Douglas operators enjoy the IAP. In [9], it is proved that the class of complex symmetric operators simultaneously have the IAP and the RAP. In [1], it is proved that the class of skew-symmetric operators have the IAP.

We are interested in determining whether several operator classes consisting of Toeplitz operators have the IAP or the RAP. Toeplitz operators are those bounded linear operators on $l^2(\mathbb{N})$ induced by infinite Toeplitz matrices of the form

$$\begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Such an operator is uniquely determined by a function ϕ in $L^\infty(\mathbb{T})$ with $\phi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ being its Fourier series, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. So we can rewrite it as T_ϕ and call ϕ the symbol of T_ϕ. Toeplitz operators are among the most studied Hilbert space operators and have found applications in a wide variety of areas such as physics, probability theory, information and control theory. The reader is referred to [2] and [10] for more about Toeplitz operators.

Given a subset E of $L^\infty(\mathbb{T})$, we denote $\mathcal{T}_E = \{T_\phi : \phi \in E\}$. Thus $\mathcal{T}_{L^\infty(\mathbb{T})}$ is exactly the class of Toeplitz operators.

Problem 14.1. *Does $\mathcal{T}_{L^\infty(\mathbb{T})}$ enjoy the IAP or the RAP?*

There are also some important subclasses of $\mathcal{T}_{L^\infty(\mathbb{T})}$ such as the class $\mathcal{T}_{H^\infty}$ of analytic Toeplitz operators and $\mathcal{T}_{C(\mathbb{T})}$, where H^∞ denotes the collection of bounded analytic functions on $|z| < 1$ and $C(\mathbb{T})$ denotes the collection of continuous functions on $\mathbb{T}$. Now it has been shown that $\mathcal{T}_{C(\mathbb{T})}$ has the IAP and $\mathcal{T}_{H^\infty}$ does not have the RAP (see [6]).

Problem 14.2. *Does $\mathcal{T}_{H^\infty}$ have the IAP?*

Problem 14.3. *Does $\mathcal{T}_{C(\mathbb{T})}$ have the RAP?*

To solve the preceding problems, it is important to solve the following problem.

Problem 14.4. *How to determine whether a Toeplitz operator is irreducible? (Zhengwei Liu)*

References

[1] Qinggang Bu, Sen Zhu, The Weyl–von Neumann theorem for skew-symmetric operators. Ann. Funct. Anal. 14 (2023), no. 2, Paper No. 43, 12 pp.

[2] Ronald G. Douglas, Banach algebra techniques in operator theory. Second edition. Graduate Texts in Mathematics, 179. Springer-Verlag, New York, 1998

[3] P. R. Halmos, Irreducible operators, Michigan Math. J. 15 (1968), 215-223.

[4] P. R. Halmos, Ten problems in Hilbert space. Bull. Amer. Math. Soc. 76 (1970), 887-933.

[5] P. R. Halmos, A Hilbert space Problem Book, GTM 19, Springer-Verlag New York, 1982.

[6] H. Huang, Y. Liu, Y. Shi, S. Zhu, Unitarily irreducible approximation of matrices, preprint

[7] Domingo A. Herrero, Chun Lan Jiang, Limits of strongly irreducible operators, and the Riesz decomposition theorem, Michigan Math. J. 37 (1990), no. 2, 283-291.

[8] You Qing Ji, Chun Lan Jiang, Small compact perturbation of strongly irreducible operators, Integral Equations Operator Theory 43 (2002), no. 4, 417-449.

[9] Ting Liu, Jiayin Zhao, Sen Zhu, Reducible and irreducible approximation of complex symmetric operators. J. Lond. Math. Soc. (2) 100 (2019), no. 1, 341-360.

[10] Nikolaï Nikolski, Toeplitz matrices and operators. Translated from the French edition by Danièle Gibbons and Greg Gibbons. Cambridge Studies in Advanced Mathematics, 182. Cambridge University Press, Cambridge, 2020

[11] Dan Voiculescu, A non-commutative Weyl-von Neumann theorem, Rev. Roumaine Math. Pures Appl. 21 (1976), no. 1, 97-113.

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